Analysis and Approaches Formulae Sheet (Standard Level and Higher Level)

Pre-Requisites

Area of Triangle	$\frac{1}{2}$ x base x height
Area of Parallelogram	base x height
Area of Rectangle	$length \times width$
Area of Trapezoid	$\frac{1}{2}$(sum of parallel sides) x height
Circumference & Area: Circle	$c = 2\pi r,\ A = \pi r^2$
Cuboid Surface area	$SA = 2xy + 2xz + 2yz$ Where $x, y,$ and z are side lengths
Cuboid Volume	$V = xyz$
Cylinder Surface Area	$SA = 2\pi rh + 2\pi r^2$ Note: Curved part: $2\pi rh$
Cylinder Volume	$V = \pi r^2 h$
Prism Volume	V =Area of cross section x height
Distance between 2 points $(x_1, y_1), (x_2, y_2)$	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Coordinates of midpoint of $(x_1, y_1), (x_2, y_2)$	$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Topic 1: Number and Algebra

Arithmetic sequence: nth term	$u_n = a + (n-1)d$ where a =first term, d= common diff
Arithmetic sequence: sum of n terms	$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$ where a=first term,d=common diff, l=last term
Geometric sequence: nth term	$u_n = ar^{n-1}$ where a =first term, r= common ratio
Geometric sequence: sum of n terms	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, r \neq 1$ where a =first term, r= common ratio
Geometric sequence: Sum to infinity	$S_\infty = \frac{a}{1-r},\ \lvert r\rvert < 1$ where a =first term, r= common ratio
Compound Interest	$FV = PV\left(1 + \frac{r}{100\,k}\right)^{kt}$ FV=future value, PV=present value t=no. of years r=nominal annual interest rate k =no. of compounding periods per year
Exponential & Logarithm Rules	• $c\log_a b \Leftrightarrow \log_a b^c$ • $\log_a b = c \Leftrightarrow a^c = b, a, b > 0, a \neq 1$ • $\log_a b + \log_a c \Leftrightarrow \log_a bc$ • $\log_a b - \log_a c \Leftrightarrow \log_a \frac{b}{c}$ • $\log_a b \Leftrightarrow \frac{\log_c b}{\log_c a}$
Binomial Theorem: integer powers	$(a+b)^n$ $= a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$
Binomial Theorem: Fractional & negative powers	$(a+b)^n$ $= a^n\left(1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \cdots\right)$
Binomial Coefficient	$\binom{n}{r} = nc_r = \frac{n!}{(n-r)!\,r!}$
Comb and Permutations	$nc_r = \frac{n!}{(n-r)!r!}, np_r = \frac{n!}{(n-r)!}$
Complex Numbers: Cartesian Form	$z = a + bi$
Complex Numbers: Modulus/Argument Form	$z = r(\cos\theta + i\sin\theta) = r\,cis\,\theta$
Complex Number: Eulers Form	$z = re^{i\theta}$
De Moivres' Theorem	$z^n = [r(\cos\theta + i\sin\theta)]^n$ $= r^n(\cos n\theta + i\sin n\theta) = r^n cis\, n\theta$

Topic 2: Functions

Straight Line: Equation (gradient means slope)	• Slope intercept form: $y = mx + c$ • General form: $ax + by + d = 0$ • Point slope form: $y - y_1 = m(x - x_1)$ Parallel⟹ same slope Perpendicular⟹ "flip fraction and change the sign" slopes multiply to make -1)
Straight Line: Gradient	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Quadratic Function: Sol to $ax^2 + bx + c = 0$	$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, a \neq 0$
Quadratic Function: Axis of Symmetry	$f(x) = x^2 + bx + c \Rightarrow x = -\frac{b}{2a}$
Quadratic Function: Discriminant	$\Delta = b^2 - 4ac$ • > 0 (2 real distinct roots) • $= 0$ (2real repeated/double roots) • < 0 (no real roots)
Exponentials and Logarithmic Functions	$a^x = e^{x\ln a}$ and $\log_a a^x = x = a^{\log_a x}$ where, $a, x > 0, a \neq 1$
Inverse	Replace $f(x)$ with y,swap x & y,solve for y
Composite	$fg(x)$ means plug $g(x)$ into $f(x)$
Odd-Even	Even: $f(-x) f(x)$, Odd: $f(-x) = -f(x)$
Transformations: $af(bx + c) + d$ "anything in a bracket does the opposite"	a=vertical stretch sf a b=horizontal stretch scale factor $\frac{1}{b}$ c=translation c unit kin x direction d=translation d units in y direction $f(-x)$=reflcn in y axis, $-f(x)$=reflcn in x axis
Sum & Product of roots of polynomial: form $\sum_{i=0}^{n} a_i x^i = 0$	Sum roots = $\frac{-a_{n-1}}{a_n}$, Product roots= $\frac{(-1)^n a_0}{a_n}$

Linear: $y = mx + c$
Domain: x∈ ℝ
Range: y∈ ℝ

Quadratic: $y = \pm a(bx + c)^2$+d
Domain: x∈ ℝ
Range: $y \geq d$ if min, $y \leq d$ if max
Domain: x∈ ℝ (Hint: power of exp can be anything, so no restriction)
Range: $y > d$ if a > 0, $y < d$ if a < 0
Asymptote: $y = d$

Logarithm: $y = a\ln(bx + c)$+d
Domain: $x > -\frac{c}{b}$
Range: y ∈ ℝ
Asymptote: $x = -\frac{c}{b}$

Root: $a\sqrt{bx + c} + d$:
Domain: $x \geq -\frac{c}{b}$
Range: $y \geq d$ if $a > 0$ and $y \leq d$ if a<0

Modulus $a\lvert bx + c\rvert + d$:
Domain: x∈ ℝ
Range: $y \geq d$ if $a > 0$ and $y \leq d$ if a <0
Note: Defn of $\lvert x\rvert = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

Rational: $\frac{ax+b}{cx+d} + e$
Domain: x∈ ℝ, $x \neq -\frac{d}{c}$
Range: y∈ ℝ, $y \neq \frac{a}{c} + e$
Asymptotes: $x = -\frac{d}{c}, y = \frac{a}{c} + e$

Trigonometric: $y = a\sin(bx + c)$+d
$y = a\cos(bx + c)$+d
Domain: x ∈ ℝ
Range: $-a + d \leq y \leq a + d$
Note: If asked to find values of a,b,c,d
a = amplitude= $\frac{\max y - \min y}{2}$
$b = \frac{2\pi}{period}$ or $\frac{360}{period}$
d = principal axis= $\frac{\max y + \min y}{2}$
c =phase shift (plug in point to find)
Trigonometric: $y = a\tan(bx + c)$+d
Domain: x ∈ ℝ, $x \neq \frac{\pi}{2} + n\pi$
Range $-\infty \leq y \leq \infty$
Inverse trig: $y = \sin^{-1}x$
Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
Inverse trig: $y = \cos^{-1}x$
Domain: $-1 \leq x \leq 1$
Range: $0 \leq x \leq \pi$
Inverse trig: $y = \tan^{-1}x$
Domain: $-\infty \leq x \leq \infty$
Range: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Topic 3: Geometry and Trigonometry

Distance between (x_1, y_1, z_1) and (x_2, y_2, z_2)	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Coordinates of midpoint of (x_1, y_1, z_1) and (x_2, y_2, z_2)	$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
Cone Surface Area	$SA = \pi rl + \pi r^2$ Note: Curved part: πrl, where l is slant length
Cone Volume	$V = \frac{1}{3}\pi r^2 h$
Sphere Surface Area	$SA = 4\pi r^2$ Note: Hemisphere= $2\pi r^2 + \pi r^2 = 3\pi r^2$
Sphere Volume	$v = \frac{4}{3}\pi r^3$ Note: Hemisphere= $\frac{2}{3}\pi r^3$
Pyramid Volume	$V = \frac{1}{3} \times base\ area \times h$
Sine Rule	Finding a side: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ Finding an angle: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
Cosine Rule	inding a side: $a^2 = b^2 + c^2 - 2bc\cos A$ Finding an angle: $A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$
Area of Triangle	$\frac{1}{2}ab\sin C$
Degrees to radians and vice versa	D to R: $\times \frac{\pi}{180}$ R to D: $\times \frac{180}{\pi}$
Length of an arc	$\frac{\theta}{360} \times 2\pi r$ (degrees) or $r\theta$ (radians)
Area of a Sector	$\frac{\theta}{360} \times \pi r^2$ (degrees) or $\frac{1}{2}r^2\theta$ (radians)
Pythagorean identity 1	$\sin^2 x + \cos^2 x = 1$
Pythagorean identity 2	$1 + \tan^2 x = \sec^2 x.$
Pythagorean identity 3	$1 + \cot^2 x = cosec^2 x$
Cofunction	$\cos x = \sin(90 - x)$ $\sin x = \cos(90 - x)$
Identity of $\tan x$	$\tan x = \frac{\sin x}{\cos x}$
Double Angle	$\sin 2x = 2\sin x\cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2\theta - 1 \Rightarrow \cos^2 x = \frac{\cos 2x+1}{2}$ $= 1 - 2\sin^2\theta \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$ $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
Reciprocal	$\sec x = \frac{1}{\cos x}, \csc x = \frac{1}{\sin x}, \cot x = \frac{1}{\tan x}$
Compound Angle	$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
Vector Form	$\boldsymbol{ai} + \boldsymbol{bj} + \boldsymbol{ck} \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
Properties (addition/subtraction, multiplication and scalar product)	$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \pm \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a \pm d \\ b \pm e \\ c \pm f \end{pmatrix}$ $\lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$ $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$
Magnitude of a vector	$\left\lvert\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\rvert = \sqrt{a^2 + b^2 + c^2}$
Unit Vector	Unit vector of $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\sqrt{a^2+b^2+c^2}}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$
Angle Between 2 vectors	$\theta = \cos^{-1}\left(\frac{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix}}{\left\lvert\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\rvert \left\lvert\begin{pmatrix} d \\ e \\ f \end{pmatrix}\right\rvert}\right)$
Vector Equation of a line	$r = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda\begin{pmatrix} d \\ e \\ f \end{pmatrix}$
Cartesian Equation of a line	$\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f}$
Parametric Form of a line	$x = a + \lambda d, y = b + \lambda e, z = c + \lambda f$
Equation of a plane	$\boldsymbol{r.n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.\boldsymbol{n}$ where n is the normal vector
Vector Equation of a plane	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda\begin{pmatrix} d \\ e \\ f \end{pmatrix} + \mu\begin{pmatrix} r \\ s \\ t \end{pmatrix}$
Cartesian Equation of a plane	$ax + by + cz = d$
Scalar Product	$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \left\lvert\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\rvert \left\lvert\begin{pmatrix} d \\ e \\ f \end{pmatrix}\right\rvert \cos\theta$ where, θ is the angle between $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$
Vector Product	$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ec \\ -(af - cd) \\ ae - bd \end{pmatrix}$ or $\left\lvert\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix}\right\rvert = \left\lvert\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\rvert \left\lvert\begin{pmatrix} d \\ e \\ f \end{pmatrix}\right\rvert \sin\theta$ where, θ is the angle between $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$
Area of a Parallelogram	$A = \left\lvert\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix}\right\rvert$ where, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ form 2 adjacent sides of a parallelogram

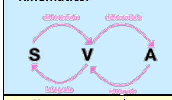
Topic 4: Statistics & Probability

Interquartile Range	IQR= $Q_3 - Q_1$
Mean	$\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$ where $n = \sum_{i=1}^{k} f_i$
Probability of event A	$P(A) = \frac{n(A)}{n(U)}$
Complementary Events	P(A')=1−P(A)
Combined Events	P(A∪B)=P(A)+P(B)−P(A ∩ B)
Mutually Exclusive Events	P(A ∩ B) = 0
Conditional	$P(A\mid B) = \frac{P(A \cap B)}{P(B)}$
Independent Events	P(A ∩ B) = P(A)P(B)
Binomial Distribution	$x \sim B(n, p)$ E(X)=Mean= np, Var(X)= $np(1-p)$
Normal Distribution	$x \sim N(\mu, \sigma^2)$ Standardised variable $z = \frac{x-\mu}{\sigma}$

Topic 4: Statistics & Probability Continued

Bayes Theorem	$P(A\mid B) = \frac{P(B\mid A)P(A)}{P(B\mid A)P(A) + P(B\mid A')P(A')}$
Variance	$\sigma^2 = \frac{\sum_{i=1}^{k} f_i(x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$
Standard Deviation	$\sigma = \sqrt{\frac{\sum_{i=1}^{k} f_i(x_i - \mu)^2}{n}} = \sqrt{\frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2}$
Linear Transformations of a random variable	$E(aX \pm b) = aE(X) \pm b$ $Var(aX \pm b) = a^2 Var(x)$
Expected Value Discrete	E(X)= $\sum xP(X = x)$
Expected Value Continuous	$Var\ (X) = \int_{-\infty}^{\infty} xf(x)dx$
Variance Discrete	E(X)= $\sum x^2 P(X = x) - \mu^2$
Variance Continuous	$Var\ (X) = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

Topic 5: Calculus

Turning/Stationary Points (Max/Min)	Solve $\frac{dy}{dx} = 0$, If $\frac{d^2y}{dx^2} > 0$ min and $\frac{d^2y}{dx^2} < 0$ max
Points of Inflection	solve $\frac{d^2y}{dx^2} = 0$
Increasing/Decreasing	Increasing: solve $\frac{dy}{dx} > 0$, decreasing: solve $\frac{dy}{dx} < 0$
Convex/Concave	concave up: solve $\frac{d^2y}{dx^2} > 0$ concave down: solve $\frac{d^2y}{dx^2} < 0$
Chain Rule	$y = g(u), u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
Product Rule	$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Area between	curve & x axis: $\int_{x=a}^{x=b} y\,dx$ curve & y axis: $\int_{y=a}^{y=b} x\,dy$ (take + answer if neg) Between 2 curves: $\int_{x=a}^{x=b}$(top curve-bottom curve)dx Remember to split up if separate areas
Kinematics: 	Distance= $\int_{t_1}^{t_2} \lvert v(t)\rvert dt$, Displacement= $\int_{t_1}^{t_2} v(t)dt$ Velocity: $\int_{t_1}^{t_2} a(t)dt$ or $\frac{ds}{dt}$ Acceleration= $\frac{dv}{dt} = \frac{d^2s}{dt^2}$
Differentiation 1st Principles	$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
Derivatives	• $x^n \Rightarrow nx^{n-1}$ • $(f(x))^n \Rightarrow n(f(x))^{n-1}f'(x)$ • $\ln(f(x)) \Rightarrow \frac{f'(x)}{f(x)}$ • $\sin f(x) \Rightarrow f'(x)\cos f(x)$ • $\cos f(x) \Rightarrow -f'(x)\sin f(x)$ • $e^{f(x)} \Rightarrow f'(x)e^{f(x)}$ • $a^{f(x)} \Rightarrow f'(x)\,a^{f(x)}\ln a$ • $\tan f(x) \Rightarrow f'(x)\sec^2 f(x)$ • $\sec f(x) \Rightarrow f'(x)\sec f(x)\tan f(x)$ • $cosec\, f(x) \Rightarrow -f'(x)cosec\, f(x)\cot f(x)$ • $\cot f(x) \Rightarrow -f'(x)csc^2 f(x)$ • $\sin^{-1} f(x) \Rightarrow \frac{f'(x)}{\sqrt{1-(f(x))^2}}$ • $\cos^{-1} f(x) \Rightarrow -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$ • $\tan^{-1} f(x)) \Rightarrow \frac{f'(x)}{1+(f(x))^2}$ • $\sec^{-1} f(x) \Rightarrow \frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$ • $cosec^{-1} f(x) \Rightarrow -\frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$ • $\cot^{-1} f(x) \Rightarrow -\frac{f'(x)}{1+(f(x))^2}$
Integrals	• $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$ • $\int \frac{1}{kx} dx = \frac{1}{k}\ln\lvert x\rvert + c$ • $\int \sin kx\,dx = -\frac{1}{k}\cos kx + c$ • $\int \cos kx\,dx = \frac{1}{k}\sin kx + c$ • $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$ • $\int a^{kx}\,dx = \frac{1}{k\ln a}a^{kx} + c$ • $\int \sec^2 kx\,dx = \frac{1}{k}\tan kx + c$ • $\int \sec kx\tan kx\,dx = \frac{1}{k}\sec kx + c$ • $\int cosec\,kx\cot kx\,dx = -\frac{1}{k}cosec\,kx + c$ • $\int cosec^2\,kx\ dx = -\frac{1}{k}\cot kx + c$ • $\int \sec kx\,dx = \frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert + c$ • $\int cosec\,kx\,dx = -\frac{1}{k}\ln\lvert cosec kx + \cot kx\rvert$+c • $\int \frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + c$ • $\int -\frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + c$ • $\int \frac{1}{a^2+(bx)^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$
Integration by parts	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
Volume of Revolution	About x axis : $V = \int_a^b \pi y^2 dx$ About y axis: $V = \int_a^b \pi x^2 dy$
Euler's method	y_{n+1} $= y_n + h \times f(x_n, y_n); x_{n+1} = x_n + h$ where h is a constant (step length)
Integrating Factor for $y' + P(x) = Q(x)$	$e^{\int P(x)dx}$
Maclaurin Series	$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$
Maclaurin Series for Special Functions	• $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ • $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ • $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ • $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ • $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$